Week 12 : (properties of determinant)

Recall : Given a $n \times n$ matrix A,

① if $n=1$, $\det A = A_{11}$

② if $n>1$, $\det A = \sum_{j=1}^{n} (-1)^{j+1} A_{j1} \cdot \det(A(j|1))$.

(It measures the singularity of A.)

$\star\star$: Thm : A is singular iff $\det A = 0$.

properties : 1) linear in columns

eg in $n=3$: $\det \begin{bmatrix} a_{11} & \lambda a_{12} + c_{12} & a_{13} \\ a_{21} & \lambda a_{22} + c_{22} & a_{23} \\ a_{31} & \lambda a_{32} + c_{32} & a_{33} \end{bmatrix}$

$= \lambda \cdot \det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} + \det \begin{bmatrix} a_{11} & c_{12} & a_{13} \\ a_{21} & c_{22} & a_{23} \\ a_{31} & c_{32} & a_{33} \end{bmatrix}$

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2) anti-symmetric in columns

eg in $n=3$: $\det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = -\det \begin{bmatrix} a_{12} & a_{11} & a_{13} \\ a_{22} & a_{21} & a_{23} \\ a_{32} & a_{31} & a_{33} \end{bmatrix}$

$= -\det \begin{bmatrix} a_{13} & a_{12} & a_{11} \\ a_{23} & a_{22} & a_{21} \\ a_{33} & a_{32} & a_{31} \end{bmatrix} = \cdots\cdots$

(If two columns are identical, then $\det A = 0$)

$\Downarrow$

special case of A = singular.

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(2)+(1): If some column is linear combination of other column, then $\det A = 0$.

eg:
$$\det\begin{bmatrix} a_{11} & a_{12} & \lambda a_{11} + \mu a_{12} \\ a_{21} & a_{22} & \lambda a_{21} + \mu a_{22} \\ a_{31} & a_{32} & \lambda a_{31} + \lambda a_{32} \end{bmatrix}$$

$$= \lambda \det\begin{bmatrix} a_{11} & a_{12} & a_{11} \\ a_{21} & a_{22} & a_{21} \\ a_{31} & a_{32} & a_{31} \end{bmatrix} + \mu \det\begin{bmatrix} a_{11} & a_{12} & a_{12} \\ a_{21} & a_{22} & a_{22} \\ a_{31} & a_{32} & a_{32} \end{bmatrix} = 0$$

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(2)+(1): det is invariant under "addition of scalar multiples of other columns".

eg:
$$\det\begin{bmatrix} \boxed{\begin{matrix} a_{11} \\ a_{21} \\ a_{31} \end{matrix}} & \begin{matrix} a_{12} \\ a_{22} \\ a_{32} \end{matrix} & \begin{matrix} a_{13} \\ a_{23} \\ a_{33} \end{matrix} \end{bmatrix} = \det\begin{bmatrix} a_{11} & a_{12} & \lambda a_{11} + a_{13} \\ a_{21} & a_{22} & \lambda a_{21} + a_{23} \\ a_{31} & a_{33} & \lambda a_{31} + a_{33} \end{bmatrix}$$

$\times\lambda$

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3): det is invariant under transpose: $\det A = \det A^t$.

eg:
$$\det\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \det\begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}.$$

(3) + (1) + (2) : det is invariant under "addition of scalar multiples of other row.

eg: $\det\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33}\end{bmatrix} = \det\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ \lambda a_{11}+a_{21} & \lambda a_{12}+a_{22} & \lambda a_{13}+a_{23} \\ a_{31} & a_{32} & a_{33}\end{bmatrix}$.

Example:

$$\det\begin{bmatrix} 1 & 9 & 7 & 7 \\ 0 & 5 & 2 & 5 \\ 1 & 9 & 8 & 0 \\ 1 & 9 & 8 & 3\end{bmatrix} \overset{-R_3+R_4}{=\!=} \det\begin{bmatrix} 1 & 9 & 7 & 7 \\ 0 & 5 & 2 & 5 \\ 1 & 9 & 8 & 0 \\ 0 & 0 & 0 & 3\end{bmatrix}$$

$$\overset{-R_1+R_3}{=\!=} \det\begin{bmatrix} 1 & 9 & 7 & 7 \\ 0 & 5 & 2 & 5 \\ 0 & 0 & 1 & -7 \\ 0 & 0 & 0 & 3\end{bmatrix} = 15$$

$$\overset{-7C_1+C_3,\ -9C_1+C_2,\ -7C_1+C_4}{=\!=} \det\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 2 & 5 \\ 0 & 0 & 1 & -7 \\ 0 & 0 & 0 & 3\end{bmatrix}$$

$$\overset{-\frac{2}{5}C_2+C_3,\ -C_2+C_4}{=\!=} \det\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 1 & -7 \\ 0 & 0 & 0 & 3\end{bmatrix}$$

$$\overset{7C_3+C_4}{=} \det\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3\end{bmatrix} = 15$$

☆☆ Thm: Suppose A, B are $n\times n$ matrix, then

$$\det(AB) = (\det A)(\det B)$$

pf: step 1: The equality holds if A = row op. matrix.

(i) $A: R_i \leftrightarrow R_j \rightsquigarrow \det A = -1$

(ii) $A: \lambda R_i,\ \lambda \in \mathbb{R} \rightsquigarrow \det A = \lambda$

(iii) $A: \lambda R_i + R_j \rightsquigarrow \det A = 1.$

$\therefore \det(AB) = \det A \cdot \det B$ by properties (1)+(2)+(3).

<u>Step 2</u>: If A = non-singular.

then $A = E_1 E_2 \cdots E_N$ where E_i = row op. matrix

$$\Rightarrow \det(AB) = \det E_1 \cdot \det E_2 \cdots \det E_N \cdot \det B.$$

$$= \det A \cdot \det B \quad \text{by induction.}$$

<u>Step 3</u>: If A = singular, then $\mathrm{RREF}(A)$ has free columns (think about free column as span of pivot)

$$\Rightarrow \det(\mathrm{RREF}(A)) = 0. \Rightarrow \det A = 0.$$

<u>Step 4</u>: If A = singular, then AB = singular, hence

$$\det AB = 0 = \det A \cdot \det B. \quad \#$$

<u>Thm</u>: $\det A \neq 0 \Leftrightarrow A$ = non-singular.

<u>pf</u>: If A = singular, then $\det A = 0$. (proved: $\Leftarrow$)

if $\det A \neq 0$, let A' be RREF of A.

then $\underbrace{\det A'}_{\neq 0} = \lambda \det A$ for some $\lambda \neq 0$.

$\Rightarrow A'$ has no free column $\Rightarrow A$ = non-singular $\#$

Summary on non-singularity : Given $n \times n$ matrix $A = [u_1, u_2 \cdots u_n]$

the followings are equivalent

① A = non-singular

② A = invertible

③ A is row equivalent to I_n

④ $\exists B$ s.t. $AB = I_n$

⑤ $\exists B$ s.t. $BA = I_n$

⑥ $\forall b \in \mathbb{R}^n$, $\exists x \in \mathbb{R}^n$ s.t. $Ax = b$

⑦ A^t = non-singular

⑧ $\mathbb{R}^n = \text{span}\{u_1, u_2, \cdots, u_n\}$

⑨ $u_1, u_2, \cdots, u_n$ are linearly indep

⑩ $u_1, u_2, \cdots, u_n$ is a basis of $\mathbb{R}^n$.

⑪ $\text{rank}(A) = n$

⑫ $\det(A) \neq 0$.

Back to eigenvalue :

Finding (possible) eigenvalues : determine $x \in \mathbb{R}$ s.t.

$$P_A(x) \triangleq \det(A - xI) = 0.$$

call : characteristic polynomial of A.

Eg: $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$, $P_A(\lambda) = \det \begin{bmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ 1 & 1 & 2-\lambda \end{bmatrix}$

$$\overset{-R_2+R_3}{=} \det\begin{bmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ 0 & -1+\lambda & 1-\lambda \end{bmatrix}$$

$$\overset{C_3+C_2}{=} \det\begin{bmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{bmatrix}$$

$$= (1-\lambda)\cdot\det\begin{bmatrix} 2-\lambda & 2 \\ 1 & 3-\lambda \end{bmatrix}$$

$$= -(1-\lambda)^2(\lambda-4).$$

$\therefore$ real roots $= 1$ OR 4.

$E_A(1) = \mathrm{Null}(A-I) = \mathrm{span}\left\{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right\}$ by RREF

$E_A(4) = \mathrm{Null}(A-4I) = \mathrm{span}\left\{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right\}.$

$\rightsquigarrow$ A = diagonalizable since $\dim(E_A(1)) + \dim(E_A(4)) = 3$.

Thm : Given $n\times n$ matrix A, $\lambda \in \mathbb{R}$, then the followings are equivalent.

① $A-\lambda I$ is singular

② λ = eigenvalue of A

③ $\det(A-\lambda I) = 0$

④ λ = real root of $P_A(x)$.

Corollary : If n = odd, then A has at least 1 eigenvalue.

☆☆☆ thm : If $A = A^t$, then A = diagonalizable !!

Thm : If A = $n\times n$ matrix which is diagonalizable and $p_A(t) = \sum_{i=0}^{n} a_i t^i$, then $\sum_{i=0}^{n} a_i A^i = 0$.

pf: diagonalizable $\Rightarrow \exists\, U$ invertible s.t.

$$U^{-1}AU = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$$

$\Rightarrow \forall m = 1, 2, \dots, n$, $U^{-1}A^m U = \mathrm{diag}(\lambda_1^m, \dots, \lambda_n^m)$

$$\therefore \quad U^{-1}\left(\sum_{m=0}^{n} a_m A^m\right)U = \sum_{m=0}^{n} a_m \left(U^{-1}AU\right)^m$$

$$= \mathrm{diag}\left(f(\lambda_1), f(\lambda_2), \dots, f(\lambda_n)\right)$$

$\because \lambda_i$ = eigenvalue of A

$\therefore f(\lambda_i) = 0 \quad \forall i = 1, 2, \dots, n$

$\Rightarrow U^{-1}\left(\sum_{i=0}^{n} a_i A^i\right)U = 0$ ← zero matrix

$\Rightarrow \sum_{i=0}^{n} a_i A^i = 0$ as a matrix ✕.